Homework Assignment 1 - Graded Problems

10.1 # 3, 4, **8**, **18**
10.2 # 4, 7, 9, **18**, 19, 25
10.3 # 2, 5, 13, 14, 15, **17**
10.4 # **8**, 13, 18, 35
10.7 # **1**, 4, 5, 8

Section 10.1

Problem 8) Solve the given boundary problem:

$$y'' + 4y = \sin x; \qquad y(0) = y(\pi) = 0.$$

Solution First solve the homogeneous problem to get:

$$y_H = c_1 \cos 2x + c_2 \sin 2x$$

Then use the method of undetermined coefficients to find the non-homogeneous solution:

Let $y_p = A \sin x$, then $y_p'' = -A \sin x$.
Plugging in you get:
$L(y_p) = y_p'' + 4y_p = -A \sin x + 4A \sin x = 3A \sin x = \sin x \Rightarrow A = \frac{1}{3}$.
So the non-homogeneous solution is $y_p = \frac{1}{3} \sin x$

Thus the general solution is: $y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \sin x$

Now to find the particular solution, use the boundary values:

$y(0) = c_1 = 0 \Rightarrow y(x) = c_2 \sin 2x + \frac{1}{3} \sin x$
$y(\pi) = 0$
Since the second boundary value is satisfied $\forall c_2 \in \mathbb{R}$, the final solution of the BVP is:

$$\boxed{y_G = c_2 \sin 2x + \frac{1}{3} \sin x}$$

Problem 18) Find the eigenvalues and eigenfunction of the following function (assume that all eigenvalues are real):

$$y'' + \lambda y = 0; \qquad y'(0) = y'(L) = 0.$$

Solution

Case 1) $\lambda > 0$

Let $\lambda = \mu^2$, then we get: $y'' + \mu^2 y = 0$.

The characteristic polynomial here is: $r^2 + \mu^2 = 0$

The roots of the equation are: $r = \pm i\mu$

General solution: $y = c_1 \cos \mu x + c_2 \sin \mu x$

Derivative of y: $y' = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$

Now impose the boundary conditions:

$y'(0) = 0 \Rightarrow y'(0) = \mu c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow y = c_1 \cos \mu x$

$y'(L) = -\mu c_1 \sin \mu L = 0$

For a non-trivial solution we must have:

$\sin \mu L = 0 \Rightarrow \mu L = n\pi, n \in \mathbb{N} \Rightarrow \mu = \frac{n\pi}{L}$

So the eigenvalues are:

$\lambda_n = \mu^2 = \left(\frac{n\pi}{L}\right)^2, n \in \mathbb{N}$

And the corresponding eigenfunctions are:

$y_n = \cos\left(\frac{n\pi x}{L}\right), n \in \mathbb{N}$

Case 2) $\lambda < 0$

Let $\lambda = -\mu^2$, then we get: $y'' - \mu^2 y = 0$.

The characteristic polynomial here is: $r^2 - \mu^2 = 0$

The roots of the equation are: $r = \mu$ (with multiplicity 2)

General solution: $y = c_1 \cosh \mu x + c_2 \sinh \mu x$

Derivative of y: $y' = \mu c_1 \sinh \mu x + \mu c_2 \cosh \mu x$

Now impose the boundary conditions:

$y'(0) = 0 \Rightarrow y'(0) = \mu c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow y = c_1 \cosh \mu x$

$y'(L) = \mu c_1 \sinh \mu L = 0$

For a non-trivial solution we must have:

$c_1 \sinh \mu L = 0 \Rightarrow c_1 = 0$

So there are no non-trivial solutions in this case.

Case 3) $\lambda = 0$

In this case, the equation reduces to: $y'' = 0$, to which the solutions are $y = c_1 x + c_2$.

So $y' = c_1$

Now impose the boundary conditions:

$y'(0) = 0 \Rightarrow y'(0) = c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow y = c_2$

$y'(L) = 0$ is OK.

So for the eigenvalue $\lambda = 0$, the corresponding eigenfunction is $y_0 = 1$.

Thus, the eigenvalues and the corresponding eigenfunctions for this problem are:

$$\boxed{\lambda_0 = 0, y_0 = 1; \lambda_n = \left(\frac{n\pi}{L}\right)^2, y_n = \cos\left(\frac{n\pi x}{L}\right), n \in \mathbb{N}}$$

Section 10.2

Problem 18) Given the function $f(x) = \begin{cases} 0 & -2 \le x \le 1 \\ x & -1 < x < 1 \\ 0 & 1 \le x < 2 \end{cases}$, (a) Sketch the graph of the given function for three periods, (b) Find the Fourier series for the given function.

Solution (a) Graph is not difficult.

(b) Our goal is to write the function in the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

We have that $L = 2$ for the formulas on page 580, so:
$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-1}^{1} x\cos\left(\frac{n\pi x}{2}\right) dx = 0$
and
$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-1}^{1} x\sin\left(\frac{n\pi x}{2}\right) dx = \int_0^1 x\sin\frac{1}{2}\pi n x\, dx$
$= -\int_0^1 \left(-\frac{2}{\pi n}\cos\frac{1}{2}\pi n x\right) dx - \frac{2}{\pi n}\cos\frac{1}{2}\pi n = \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2} - \left(\frac{2}{n\pi}\right)\cos\frac{n\pi}{2}$

So the Fourier series expansion of this function is given by:

$$\boxed{f(x) = \sum_{n=1}^{\infty} \left(\left(\left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2} - \left(\frac{2}{n\pi}\right)\cos\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2}\right)\right)}$$

Section 10.3

Problem 17) Assuming that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}),$$

show formally that

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Solution Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}).$$

Squaring both sides of the equation we get:

$$|f(x)|^2 = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}(a_n^2 \cos^2 \frac{n\pi x}{L} + b_n^2 \sin^2 \frac{n\pi x}{L})$$
$$+a_0 \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) + \sum_{m \neq n}(c_{mn} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L}).$$

Then we integrate both sides of the equation from $-L$ to L, and use the orthogonality of sine and cosine to get:

$$\int_{-L}^{L} |f(x)|^2 \, dx = \int_{-L}^{L} \frac{a_0^2}{4} dx + \sum_{n=1}^{\infty}(\int_{-L}^{L} a_n^2 \cos^2 \frac{n\pi x}{L} dx + \int_{-L}^{L} b_n^2 \sin^2 \frac{n\pi x}{L} dx)$$
$$= \frac{a_0^2}{2} L + \sum_{n=1}^{\infty} \left(a_n^2 L + b_n^2 L\right)$$

Then, dividing by L yields the desired result:

$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right).$$

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Section 10.4

Problem 8) The function $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ x - 1 & 1 \leq x < 2 \end{cases}$, is defined on an interval of length L. Sketch the graphs of the even and odd extentions of f of period $2L$.

Solution Easy.

Problem 15) Find the required Fourier series for $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$, cosine series of period 4, and sketch the graph of the function to which the series converges over three periods.

Solution -
This means that here we can use the information on page pg. 596.
So we have that $L = 2$.
Then for $n > 0$:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^1 0\cdot\cos\frac{n\pi x}{2}dx + \int_1^2 1\cdot\cos\frac{n\pi x}{2}dx$$

$$= 0 + \int_1^2 1\cdot\cos\frac{n\pi x}{2}dx = \frac{2}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left(0 - \sin\frac{n\pi}{2}\right) = \frac{2}{\pi}\left(\frac{(-1)^{n-1}}{2n-1}\right)$$

And for $n = 0$:

$$a_0 = \int_1^2 1\cdot\cos\frac{0\pi x}{2}dx = \int_1^2 1\cdot\cos 0dx = \int_1^2 1\cdot 1dx = \int_1^2 1dx = 1$$

Thus since this is a cosine series we have that the Fourier Expansion is given by:

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{2n-1}\right)\cos\frac{n\pi x}{2}}$$

Section 10.7

Problem 1a) Consider an elastic string of length L whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x,0) = \begin{cases} \frac{2x}{L} & 0 \le x \le \frac{L}{2} \\ \frac{2(L-x)}{L} & \frac{L}{2} < x \le L \end{cases}$. Find the displacement $u(x,t)$ for the given initial position.

Solution Since the initial velocity is zero, the solution is given by $u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$, where $c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$.

So for this problem we get that:

$$c_n = \frac{2}{L}\left(\int_0^{\frac{L}{2}} \frac{2x}{L}\sin\frac{n\pi x}{L}dx + \int_{\frac{L}{2}}^{L} \frac{2(L-x)}{L}\sin\frac{n\pi x}{L}dx\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

And thus the solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}}$$